Acoustics in the Lagrange picture: an application to the Rayleigh radiation pressure

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At the undergraduate level, most lectures and textbooks on hydrodynamics make use of the so-called Euler picture, where the pressure, temperature and velocity of the fluid are treated as continuous fields defined by the value they take at each point of the reference frame the fluid moves in. There nevertheless exists another possible description of the movement which consists in labelling the fluid elements themselves, and keeping this labelling in the course of the motion. This so-called Lagrange picture is scarcely taught for it often introducess more complicated mathematics, as soon as a three-dimensional geometry is considered. Yet it is actually more intuitive than the Euler picture. In this paper, we illustrate the point in the example of the Rayleigh acoustic radiation pressure. An improved physical insight ensues, which is of interest to students graduating in acoustics.

I. INTRODUCTION

Fluid mechanics is often presented[1,2] and taught, at least at the undergraduate level, in a way quite similar to electrodynamics. To put it briefly, a (usually Galilean) reference frame is chosen. At a given point $\vec{r}$ of this frame, and at time t, the physical state of the fluid is described by a set of functions: mass density ρ, pressure P, fluid velocity $\vec{v}$ (with respect to the frame), temperature T, and so on. So one deals with a set of (coupled) continuous fields $\rho(\vec{r},t)$, $P(\vec{r},t)$, $\vec{v}(\vec{r},t)$, $T(\vec{r},t)$, *etc.* For instance, in the absence of any external force (gravity or else), the movement of an inviscid[19] fluid is governed by the well-known Euler equation,

$$\rho(\vec{r},t)\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\overrightarrow{\mathrm{grad}})\,\vec{v}\right) = -\,\overrightarrow{\mathrm{grad}}\,P. \tag{1}$$

This description of the fluid motion will be referred to in the present paper as the Euler picture. Such a picture has many advantages. First, it is convenient to describe the dynamics of the fluid by means of *local* equations coupling fields, exactly as in electrodynamics. The Euler equation (1) is an example of such a local description. Second, the Euler picture is particularly well suited to situations in which the fluid really *flows*: when studying the stream of a river passing under some bridge, we are interested in the very behaviour of the water under this bridge at a time t, whatever the origin or the past behaviour of this water.

Nevertheless, the Euler picture has a few drawbacks. First, in equation (1), the left-hand side is obviously nonlinear in field $\vec{v}$, due to the $(\vec{v}\cdot\overrightarrow{\mathrm{grad}})\,\vec{v}$ term. A second drawback of the Euler picture shows up when free boundary conditions between two fluids must be imposed. Let us consider the example illustrated in figure 1: two different fluids − say 1 and 2 − are separated *at rest* by the (infinite) plane at $x = 0$. Consider a plane pressure wave propagating from $x = -\infty$ in fluid 1 towards the boundary. As is well known, this incident wave splits at the interface into two parts: a reflected wave, travelling back to $x = -\infty$ through medium 1, and a transmitted wave, travelling towards $x = +\infty$ through medium 2. It is a time-honoured undergraduate level exercise to determine the reflection and transmission coefficients at the interface. In principle, the answer is easy: the continuity of pressure (due to the finite acceleration of a fluid element),

$$P_1(\text{interface}) = P_2(\text{interface}), \tag{2a}$$

and of velocitiy (due to the absence of a gap),

$$\vec{v}_1(\text{interface}) = \vec{v}_2(\text{interface}), \tag{2b}$$

provides two equations enabling us to calculate both coefficients. The above reasoning is undoubtedly correct, but raises a non-trivial difficulty: *where* is the interface? At $x = 0$? Certainly not, since the interface itself moves back

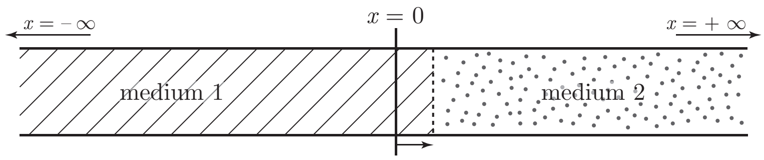

FIG. 1: Medium 1 and medium 2 are separated at rest by the plane $x = 0$. When a pressure wave propagates, the interface does not remain at $x = 0$, but moves back and forth on either side of the plane $x = 0$. In the Euler picture, locating the interface at $x = 0$ appears as a zero-order approximation.

and forth, due to the wave motion. As a matter of fact, the $x = 0$ plane spends half the time in medium 1 and half the time in medium 2. Of course, locating the interface at $x = 0$ is the best *approximation*, and it leads to the correct values of the reflection and transmission coefficients, but it should be regarded only as a zero-order approximation.

In fact, the difficulties of the Euler picture mentioned above can (up to a point) be overcome using a different framework, known as the Lagrange picture[3–5].[20] It is precisely the aim of section II to sketch the main features of the Lagrange picture which is hardly taught in academic courses and scarcely used when studying acoustic wave propagation in fluids. The reason is that the Lagrange picture brings in some involved mathematics like tensor calculus and differential geometry as soon as one considers a three-dimensional propagation. Consequently, for the sake of simplicity, we shall restrict ourselves to the one-dimensional case and show in subsection II A that, in the Lagrange picture, linearity or nonlinearity is a purely thermodynamic issue. Subsection II B is devoted to the linear thermodynamic response and the attendant picture of sound propagation. Section III deals with the nonlinearity of the thermodynamic response and tackles the cumbersome problem of the Rayleigh acoustic radiation pressure, which can be given a simpler solution in the Lagrange picture with greater physical insight.

II. AN OUTLINE OF THE LAGRANGE PICTURE

A. What is it all about?

Contrary to the Euler picture, which labels the geometric points of the reference frame disregarding the origin of the fluid elements passing through these points at time t, the Lagrange picture labels the fluid elements disregarding the position they occupy at time t. More specifically, consider a fluid at some time t_0. We denote by $\vec{r}_0$ the fluid element that occurs to stand at point $\vec{r}_0$ of the reference frame at time t_0. We shall henceforth keep this label $\vec{r}_0$ to denote this fluid element, whatever its later position. Thus, at time $t > t_0$, the fluid element $\vec{r}_0$ will be found at some point $\vec{r}$ given by

$$\vec{r}(\vec{r}_0, t) = \vec{r}_0 + \vec{u}(\vec{r}_0, t), \tag{3}$$

where $\vec{u}(\vec{r}_0, t)$ is the displacement undergone by the fluid element $\vec{r}_0$ between times t_0 and t (see fig. 2). The physical state of the fluid is still described by a set of continuous fields: mass density, pressure, velocity, temperature, *etc.* The correspondence between both pictures is very simple. With superscripts $\mathcal{E}$ and $\mathcal{L}$ respectively standing for "Euler" and "Lagrange", and quantity A standing for whichever parameter ρ, P, $\vec{v}$, T, *etc.*, we have

$$A^{\mathcal{E}}\big(\vec{r}(\vec{r}_0, t), t\big) = A^{\mathcal{L}}(\vec{r}_0, t), \tag{4}$$

with $\vec{r}(\vec{r}_0, t)$ given by (3). Concretely, the above equation means that $A^{\mathcal{L}}(\vec{r}_0, t)$ denotes the actual value of parameter A taken at time t by the fluid element labelled $\vec{r}_0$ which is currently at point $\vec{r}_0 + \vec{u}(\vec{r}_0, t)$ of the reference frame, *i.e.* $A^{\mathcal{L}}(\vec{r}_0, t)$ mathematically coincides with to $A^{\mathcal{E}}\big(\vec{r}(\vec{r}_0, t), t\big)$.

In appendix A, the Lagrange picture is applied to the calculation of the reflection and transmission coefficients at an interface between two fluids. It is shown that the Lagrange picture offers several advantages from a *technical* point of view. To begin with, this picture rids us of the formal nonlinearity associated with the $(\vec{v} \cdot \overrightarrow{\mathrm{grad}})\, \vec{v}$ term on the left-hand side of the Euler equation (1). This enables us to recognize genuine nonlinearities, thereby allowing a perturbative resolution of the field equations. We shall take advantage of such a simplification in section III, when dealing with the Rayleigh acoustic radiation pressure.

We focus on the simplest situation one may have to face: the one-dimensional problem. Let us therefore consider a fluid at rest occupying a cylindrical volume with axis Ox_0 and cross-sectional area S (fig. 3a), at equilibrium pressure

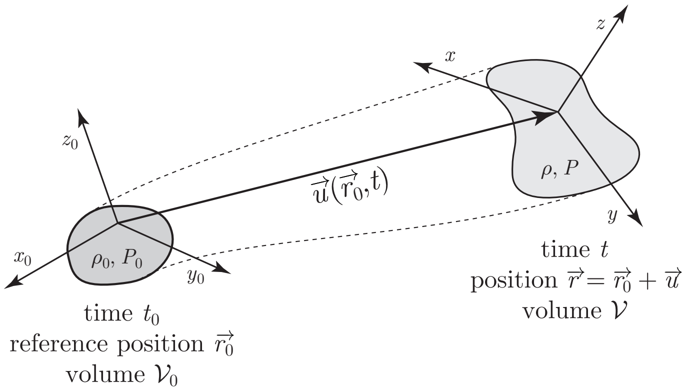

FIG. 2: An element of fluid with reference position $\vec{r}_0$ is displaced and deformed in the course of time. In the Lagrange picture, this fluid element is labelled $\vec{r}_0$ and keeps this label throughout its motion.

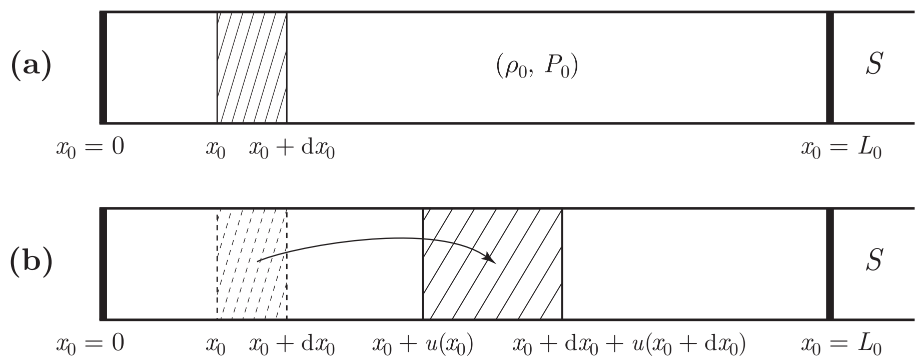

FIG. 3: (a). The fluid at rest, with equilibrium mass density ρ_0 and pressure P_0. (b). The fluid at time t: both ends, labelled $x_0 = 0$ and $x_0 = L$, are made up of pistons that are provisionally supposed fixed.

P_0 and mass density ρ_0. Both ends, labelled $x_0 = 0$ and $x_0 = L$, are bounded by pistons that are provisionally supposed to be fixed. As displayed in fig. 3, the slice of fluid located between faces x_0 and $x_0 + \mathrm{d}x_0$ has mass $\rho_0 S \mathrm{d}x_0$, where ρ_0 is the equilibrium mass density. At time t, its thickness is

$$\big(x_0 + \mathrm{d}x_0 + u(x_0 + \mathrm{d}x_0, t)\big) - \big(x_0 + u(x_0, t)\big) = \left(1 + \frac{\partial u}{\partial x_0}\right) \mathrm{d}x_0, \tag{5a}$$

so that its mass density is just

$$\rho(x_0, t) = \frac{\rho_0}{1 + \dfrac{\partial u}{\partial x_0}}. \tag{5b}$$

In the Lagrange picture, the pressure forces undergone by the slice of fluid are respectively $SP_0(x_0, t)$ (left end) and $-\, SP(x_0 + \mathrm{d}x_0, t)$ (right end). Applying Newton's Second Law to the slice, we obtain

$$\rho_0 \frac{\partial^{\mathcal{L}2} u}{\partial t^2} = -\, \frac{\partial P}{\partial x_0}, \tag{6}$$

where superscript $\mathcal{L}$ on the left-hand side recalls that the time derivative is understood at constant x_0 (even if the current position at time t of the face labelled "x_0" is miles away from the point with abscissa x_0 of the reference frame). Equation (6) deserves two comments: (i) it is exact (no approximation was made); (ii) it is strictly linear in displacement u or in velocity $v = \frac{\partial^{\mathcal{L}} u}{\partial t}$. If we now want to get a closed-form propagation equation, we have to connect the pressure $P(x_0, t)$ with the expansion factor $\frac{\partial u}{\partial x_0}$ or equivalently the mass density ρ. This connection involves thermodynamics. Throughout the present article, we shall assume, for the sake of simplicity, that any transformation undergone by the fluid is *isentropic*. In the framework of the Lagrange picture, this means that the entropy of any fluid slice $[x_0, x_0 + \mathrm{d}x_0]$ is, at any time, equal to its equilibrium value. So, in the course of the motion, the pressure $P(x_0, t)$ can be expressed as a function of the sole[21] mass density $\rho(x_0, t)$. In the isentropic relation of state $P = P(\rho)$, let us expand the extra pressure $P(x_0, t) - P_0$ in increasing powers of $(\rho(x_0, t) - \rho_0)/\rho_0 = \big(1 + \frac{\partial u}{\partial x_0}\big)^{-1}$:

$$P(x_0, t) - P_0 = -\kappa_1 \left(\frac{\partial u}{\partial x_0}\right) + \frac{1}{2}\kappa_2 \left(\frac{\partial u}{\partial x_0}\right)^2 + \cdots , \tag{7}$$

where the compressibility $\kappa_1 > 0$, due to the Second Law of thermodynamics. Combining the mechanical equation (6), in which the superscript $\mathcal{L}$ for "Lagrange" is henceforth omitted, with the above thermodynamic relation (7), we get the sound propagation equation

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = \kappa_1 \frac{\partial^2 u}{\partial x_0^2} \left(1 - \frac{\kappa_2}{\kappa_1} \frac{\partial u}{\partial x_0} + \cdots \right). \tag{8}$$

The above equation is nonlinear in displacement u, its nonlinearity originating exclusively in the κ_2, κ_3, *etc.* terms in the thermodynamic expansion (7).

B. The linear thermodynamic response and the propagation of sound

In this subsection, we linearize equation (7), *i.e.* we take $\kappa_2 = \kappa_3 = \cdots = 0$. The propagation equation (8) also becomes linear, and reads

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x_0^2}, \tag{9a}$$

with

$$c = \sqrt{\frac{\kappa_1}{\rho_0}}, \tag{9b}$$

which is the usual d'Alembert wave equation for sound propagation. Let us look for the associated eigenmodes, *i.e.* the monochromatic solutions of (9a). Owing to the boundary conditions we have chosen, they necessarily read

$$u_n(x_0, t) = \Re\big\{A_n \sin(k_n x_0)\, \mathrm{e}^{-\mathrm{i}\omega_n t}\big\}, \tag{10a}$$

with $\Re$ denoting the real part. In (10a),

$$\omega_n = c k_n, \qquad k_n = \frac{n\pi}{L_0} \quad (n = 1, 2, \cdots), \tag{10b}$$

and A_n is a complex amplitude. Note that, insofar as the linearization of the *thermodynamic* relation (7) is relevant, the above solution is *exact*, contrary to the solution generally proposed in the framework of the Euler picture, which also involves neglecting the $(\vec{v} \cdot \overrightarrow{\mathrm{grad}})\, \vec{v}$ term.[22]

Let us now determine the overall acoustic energy associated with the wave, *i.e.* the variation (with respect to the rest state) of the total energy of all slices $[x_0, x_0 + \mathrm{d}x_0]$. Since there is neither a heat exchange between neighbouring slices nor an external force, we just have to determine the work done by the pressure force to drive each fluid slice from its equilibrium state to its current state at time t. For the $[x_0, x_0 + \mathrm{d}x_0]$ slice, the work is exactly

$$\begin{aligned} \mathrm{d}E &= \int_0^t \mathrm{d}t' \left[- S P(x_0 + \mathrm{d}x_0, t') \frac{\partial u(x_0 + \mathrm{d}x_0, t')}{\partial t'} + S P(x_0, t') \frac{\partial u(x_0, t')}{\partial t'} \right] \\ &= - S \mathrm{d}x_0 \int_0^t \mathrm{d}t' \frac{\partial}{\partial x_0}\left(P \frac{\partial u}{\partial t'} \right) = - S \mathrm{d}x_0 \int_0^t \mathrm{d}t' \left[\frac{\partial P}{\partial x_0} \frac{\partial u}{\partial t'} + P \frac{\partial^2 u}{\partial x_0 \partial t'} \right]. \end{aligned} \tag{11a}$$

Owing to (6) and to the linearized version of (7), the above equation becomes

$$\mathrm{d}E = S \mathrm{d}x_0 \left[\frac{1}{2}\rho_0 \left(\frac{\partial u(x_0, t)}{\partial t} \right)^2 - P_0 \frac{\partial u(x_0, t)}{\partial x_0} + \frac{1}{2}\kappa_1 \left(\frac{\partial u(x_0, t)}{\partial x_0} \right)^2 \right]. \tag{11b}$$

Integrating over the whole fluid and accounting the boundary conditions, we finally get the overall acoustic energy E, which is a constant of the movement:

$$E = \frac{1}{2}\rho_0 S \int_0^{L_0} \mathrm{d}x_0 \left[\left(\frac{\partial u}{\partial t} \right)^2 + c^2 \left(\frac{\partial u}{\partial x_0} \right)^2 \right]. \tag{11c}$$

Now, since any solution $u(x_0, t)$ of the wave equation (9a) is a linear combination of eigenmodes of the type (10a), the above energy E may also be written, according to Parseval theorem,

$$E = \frac{1}{4}\rho_0 S L_0 \sum_{n=1}^{\infty} |A_n|^2 \omega_n^2. \tag{11d}$$

We complete these results with the following thought experiment. Suppose that, while a given eigenmode (say, n) is established in the cylindrical cavity bounded by the two pistons displayed in fig. 3, we *slowly* move the piston located at the end labelled "$x_0 = L_0$" at, say, a constant velocity V. By "slowly", we mean "adiabatically in the Ehrenfest sense". In this connection, let us define the effective number N_n of quanta in mode n such that $N_n \hbar\omega_n = \frac{1}{4}\rho_0 S L_0 |A_n|^2 \omega_n^2$, so that $E = \sum_{n=1}^{\infty} N_n \hbar\omega_n$.[23] We have discussed at some length this issue in a former paper[6], and shown that, in the course of such an adiabatic parametric excitation of the system, the number N_n of quanta is conserved. Let us recall that, in the Lagrange picture, the label "$x_0 = L_0$" of the fluid in contact with the moving piston remains unchanged, although the total length of the fluid column is obviously $L(t) = L_0 + Vt$. In this respect, it is convenient to split the displacement $u(x_0, t)$ into two parts, and let

$$u(x_0, t) = \frac{x_0}{L_0} V t + w(x_0, t). \tag{12}$$

The first term on the right-hand side is the displacement of the slice labelled x_0, associated with a quasistatic expansion (or compression, according to the sign of V) of the fluid. The second term is the extra displacement of the fluid slice due to the acoustic wave. Observe that the boundary conditions for $w(x_0, t)$ are the same as for $u(x_0, t)$: $w(x_0 = 0, t) = w(x_0 = L_0, t) = 0$. Now, let us rewrite the wave equation (9a) in terms of w instead of u. From (12), we get

$$\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x_0^2}, \tag{13}$$

so that w and u are governed by the same equation. Solution (10a) is consequently unchanged: surprising though it may be, the motion of the piston has strictly no influence upon eigenmode n. In particular, no frequency shift[24] occurs: the wave number $k_n = \frac{n\pi}{L_0}$ as well as the angular frequency $\omega_n = c k_n$ keep their initial values even if $L(t)$ happens to become twice (or half) its initial value L_0. Moreover, from equation (11d) one concludes that, since neither the quanta number N_n (Ehrenfest adiabaticity) nor the angular frequency ω_n are modified, the acoustic energy is unchanged. To move the piston, the operator has of course to account for the quasistatic variation of the instantaneous equilibrium pressure $P_{\mathrm{eq}}(t)$ of the fluid:

$$P(x_0, t) - P_0 = -\kappa_1 \frac{\partial u}{\partial x_0} = -\kappa_1 \left(\frac{Vt}{L_0} + \frac{\partial w}{\partial x_0} \right) \rightsquigarrow P(x_0, t) - P_{\mathrm{eq}}(t) = -\kappa_1 \frac{\partial w}{\partial x_0}, \tag{14a}$$

with

$$P_{\mathrm{eq}}(t) = P_0 - \kappa_1 \frac{Vt}{L_0}, \tag{14b}$$

but he has no extra work to supply, associated with the acoustic wave itself. Before finishing with our thought experiment, we should emphasize one point: although equation (13) holds whatever the value of velocity V, the Ehrenfest adiabaticity is required during the initial acceleration of the piston, from $V = 0$ to its final speed.

Another interesting point is: what happens when our acoustic wave meets an interface, as illustrated in fig. 1? This point has been raised in the introduction (see equations (2) and the attendant text). The Lagrange picture is shown in appendix A to provide a straightforward answer to this question.

The propagation of longitudinal expansion/compression waves through a mass-distributed spring (*e.g.* such as those designed as decorative objects or toys for children) is well described by the equations of this subsection, but the ideal fluid considered in the calculations does not exist. In the next subsection, we consider a more realistic approximation of the thermodynamic relation (7), better suited to real fluids.

III. TAKING NONLINEARITY INTO ACCOUNT: THE RAYLEIGH RADIATION PRESSURE

Let us now retain the second-order term in (7):

$$P(x_0,t) - P_0 = -\kappa_1\left(\frac{\partial u}{\partial x_0}\right) + \frac{1}{2}\kappa_2\left(\frac{\partial u}{\partial x_0}\right)^2. \tag{15}$$

Accounting for the nonlinearity κ_2 has the following consequence. Suppose that we move the right piston (labelled "$x_0 = L_0$") by an amount δL, and consider the motion of the fluid with respect to this new equilibrium position. Then we have, replacing Vt with δL in (12),

$$u(x_0,t) = \frac{x_0}{L_0}\delta L + w(x_0,t), \tag{16a}$$

so that (15) becomes

$$P(x_0,t) - P_{\rm eq} = -\kappa_1'\left(\frac{\partial w}{\partial x_0}\right) + \frac{1}{2}\kappa_2\left(\frac{\partial w}{\partial x_0}\right)^2, \tag{16b}$$

with

$$P_{\rm eq} = P_0 - \kappa_1\frac{\delta L}{L_0} + \frac{1}{2}\kappa_2\left(\frac{\delta L}{L_0}\right)^2, \tag{16c}$$

$$\kappa_1' = \kappa_1\left(1 - \frac{\kappa_2}{\kappa_1}\frac{\delta L}{L_0}\right). \tag{16d}$$

Expressions (16b) and (15) are alike. They differ in the equilibrium pressure, as was already noticed in the linear case ((16c) is just the generalization of (14b) with Vt replaced by δL). They differ also in the *linear* compressibility coefficient κ_1 being changed into κ_1', due to the nonzero value of κ_2. This change of the compressibility causes a frequency shift when moving piston L_0, as well as an acoustic radiation pressure. We shall come back to this nomenclature later and show that the frequency shift and the radiation pressure are inherently entangled and, so to say, *consubstantial*.

How can we solve for $u(x_0,t)$ obeying the nonlinear propagation equation (8)? Unless an exact solution can be found, a good approach is the perturbation method, provided that the condition $|\frac{\partial u}{\partial x_0}| \ll 1$ is fulfilled. For the sake of simplicity, let us start from an eigenmode of the linearized wave equation, say (see (10a))

$$u^{(1)}(x_0,t) = A\sin(kx_0)\cos(\omega t - \varphi), \tag{17a}$$

where we have dropped the eigenmode index n. While $u^{(1)}(x_0,t)$ is not a solution of the non linear wave equation, the exact solution having $u^{(1)}$ as its linear approximation can be expanded in increasing powers of amplitude A:

$$u(x_0,t) = u^{(1)}(x_0,t) + u^{(2)}(x_0,t) + \cdots, \tag{17b}$$

where $u^{(i)} \sim A^i$. Let us look for $u^{(2)}(x_0,t)$. Using (8) and (9b), we have

$$\frac{1}{c^2}\frac{\partial^2 u^{(2)}}{\partial t^2} - \frac{\partial^2 u^{(2)}}{\partial x_0^2} = -\frac{\kappa_2}{\kappa_1}\frac{\partial^2 u^{(1)}}{\partial x_0^2}\frac{\partial u^{(1)}}{\partial x_0}. \tag{18a}$$

The above equation means that the first-order solution $u^{(1)}$ acts as a source term for the second-order displacement $u^{(2)}$. From (17a), this source term is:

$$-\frac{\kappa_2}{\kappa_1}\frac{\partial^2 u^{(1)}}{\partial x_0^2}\frac{\partial u^{(1)}}{\partial x_0} = \frac{\kappa_2}{\kappa_1}\frac{A^2k^3}{2}\sin(2k_0x_0)\frac{1+\cos(2\omega t - 2\varphi)}{2}. \tag{18b}$$

As a result, solution $u^{(2)}$ is the sum of two contributions: one is static and the other is oscillating at the angular frequency 2ω. Let us focus on the former contribution. Accounting for the boundary conditions $u^{(2)}(x_0 = 0, t) = u^{(2)}(x_0 = L_0, t) = 0$, it is

$$u_{\rm s}^{(2)}(x_0) = \frac{\kappa_2}{\kappa_1}\frac{A^2k}{16}\sin(2kx_0), \tag{19}$$

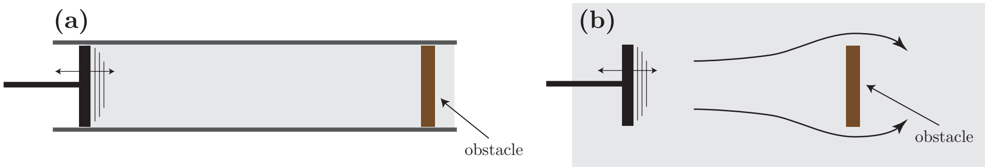

FIG. 4: Two kinds of acoustic radiation pressure, according to the geometry. (a) Rayleigh configuration: the fluid that propagates the acoustic wave is bounded by the obstacle. (b) Langevin configuration: the wave can skirt around the obstacle.

with index "s" standing for "static".

Now it is interesting to calculate, up to the second order in amplitude A, the static extra pressure $P_{\mathrm{s}}^{[2]}-P_0$ associated with the acoustic mode. From (15), we get

$$P^{[2]}(x_0,t)-P_0 = -\kappa_1\left(\frac{\partial u^{(1)}}{\partial x_0}+\frac{\partial u^{(2)}}{\partial x_0}\right)+\frac{1}{2}\kappa_2\left(\frac{\partial u^{(1)}}{\partial x_0}\right)^2. \tag{20}$$

The first-order extra pressure term $-\kappa_1\frac{\partial u^{(1)}}{\partial x_0}$ oscillates at the angular frequency ω, and consequently averages to zero over time. The second-order extra pressure term is the sum of a static and a 2ω-oscillating contribution. Focussing on the former contribution, we find, after all calculations have been carried out,

$$P_{\mathrm{s}}^{[2]}-P_0=\frac{1}{8}\kappa_2 A^2 k^2. \tag{21}$$

Quantity $P_{\mathrm{s}}^{[2]}-P_0$ is *homogeneous* ($P_{\mathrm{s}}^{[2]}$ does not depend on x_0), as expected from a static extra pressure (otherwise it would entail a permanent flow). This static extra pressure is known as the Rayleigh radiation pressure. At this juncture, a comparison with electromagnetic waves is of interest.

The existence of radiation pressure exerted by an electromagnetic wave onto some encountered obstacle is well known and can be easily figured out even at the undergraduate level. The electric field $\vec{E}$ of the wave moves electric charges in the obstacle (free carriers in a conductor, bound charges in a dielectric), resulting in an electric current density $\vec{\jmath}$. Then, the magnetic field $\vec{B}$ of the wave exerts the Lorentz force density $\vec{\jmath}\times\vec{B}$ onto the charges and the latter force is transmitted to the obstacle's lattice. The calculation of the overall force undergone by the obstacle is easy in the one-dimensional case (plane wave encountering a plane obstacle). We owe to Maxwell the first prediction of this effect in 1871[7]. This phenomenon can also be looked at in a quantum framework: the incident electromagnetic wave is regarded as a flux of photons, each photon with energy $h\nu$ carrying a mechanical momentum $h\nu/c$. When the wave interacts with matter, the latter momentum (or a part of it) is transferred to the lattice and the classical result is recovered[8]. Such a radiation pressure was experimentally observed in 1899 by P. N. Lebedev[9] using a device analogous to the Nichols radiometer.

In analogy with electrodynamics, J.W. Rayleigh introduced the acoustic radiation pressure as early as 1902[10,11]. It was observed by W. Altberg in 1903[12]. Nevertheless, although an acoustic stress tensor can be defined, which is *prima facie* analogous to the Maxwell tensor, the comparison stops there: on the one hand, no microscopic mechanism providing us with an acoustic force density analogous to the Lorentz force can be put forward; on the other hand, the photon has no real acoustic counterpart: the phonon is but a quasiparticle carrying no mechanical momentum. Moreover, while there exists only one electromagnetic radiation pressure, there are at least two kinds of acoustic radiation pressure, according as the fluid in which the acoustic wave propagates is bounded (the Rayleigh configuration, illustrated in fig. 4a) or is free to skirt around the obstacle (the Langevin configuration, illustrated in fig. 4b). Since the beginning of the twentieth century, a wealth of studies have been devoted to this rather puzzling and little understood issue[13–17].

It is clear from figure 4 that the Rayleigh configuration can be implemented in a one-dimensional geometry. On the contrary, the Langevin configuration involves a two- or three-dimensional geometry. For the sake of a pedagogic account, we have focussed in this paper on the simpler one-dimensional case. Moreover we argue that the calculation is easier in the framework of the Lagrange picture.

Let us go back to formula (21). It is noteworthy that $P_{\mathrm{s}}^{[2]}-P_0$ is proportional to coefficient κ_2, and thus owes its very existence to the nonlinearity of the thermodynamic relation (15): this is the reason why we could not find such an extra pressure in the linear framework of subsection II B. A simple relation exists between the Rayleigh radiation pressure and the (Lagrange) acoustic energy density $\mathcal{E}$ that can be derived from (11d):

$$\mathcal{E} = \frac{E}{SL_0} = \frac{1}{4}\rho_0 A^2\omega^2 = \frac{1}{4}\kappa_1 A^2 k^2. \tag{22a}$$

According to (21), the relation is

$$P_{\rm s}^{[2]} - P_0 = \frac{1}{2}\frac{\kappa_2}{\kappa_1}\mathcal{E}. \tag{22b}$$

Lastly, let us show how deeply the radiation pressure and the frequency shift are entangled. Consider again the thought experiment discussed in subsection II B, in which we slowly move the piston at the end labelled "$x_0 = L_0$". Splitting the displacement u according to (12), we are led to modify equations (14a-b) according to (16b-c), *i.e.*

$$P(x_0,t) - P_{\rm eq}(t) = -\kappa_1\left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right)\frac{\partial w}{\partial x_0} + \frac{1}{2}\kappa_2\left(\frac{\partial w}{\partial x_0}\right)^2, \tag{23a}$$

with

$$P_{\rm eq}(t) = P_0 - \kappa_1\frac{Vt}{L_0} + \frac{1}{2}\kappa_2\left(\frac{Vt}{L_0}\right)^2. \tag{23b}$$

As was already mentioned, the important point is that the *linear* term $-\kappa_1\frac{\partial w}{\partial x_0}$ is changed into $-\kappa_1'\frac{\partial w}{\partial x_0}$, due to the variation $\delta L = Vt$ of the length of the cavity:

$$\kappa_1' = \kappa_1\left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right), \tag{24}$$

in agreement with (16d). Consequently, the wave equation governing w becomes

$$\rho_0\frac{\partial^2 w}{\partial t^2} = \kappa_1'\frac{\partial^2 w}{\partial x_0^2}\left(1 - \frac{\kappa_2}{\kappa_1}\frac{\partial w}{\partial x_0}\right), \tag{25}$$

which is the *same* equation as for u except that κ_1 is replaced by κ_1' (compare for instance with (8)). Linearizing equation (25), we find a wave equation with a *modified* speed of sound c' given by

$$c'^2 = \frac{\kappa_1'}{\rho_0} = c^2\left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right). \tag{26a}$$

This modification of the speed of sound, associated with an unchanged[25] wavevector $k_n = \frac{n\pi}{L_0}$, shifts the angular frequency:

$$\omega'^2 = \omega^2\left(1 - \frac{\kappa_2}{\kappa_1}\frac{Vt}{L_0}\right). \tag{26b}$$

Observe in passing that the amplitude A of mode n is changed, too. Nevertheless, since the piston is moved adiabatically (in the Ehrenfest sense), we have (see (11d) and the discussion thereafter)

$$A'^2\omega' = A^2\omega. \tag{26c}$$

As a consequence, the change in the acoustic energy of the wave is

$$\delta E = \frac{1}{4}\rho_0 S L_0 A^2\omega\delta\omega. \tag{27a}$$

Using (26b), (9b), (10b) and (21), this is tantamount to

$$\delta E = -\frac{1}{8}\kappa_2 A^2 k^2 S\delta L = -(P_{\rm s}^{[2]} - P_0)\delta\mathcal{V}, \tag{27b}$$

which shows that δE is the work the operator has to supply in order to vary the volume of the cavity by an amount $\delta\mathcal{V} = S\delta L$. The acoustic pressure present is $-\delta E/\delta\mathcal{V}$. This is in line with the usual definition of a pressure, given that the transformation is isentropic.

In appendix B, it is shown that the radiation pressure is related to the static relative expansion through the compressibility κ_1. It is also shown that the present description of the radiation pressure carries over to solid state physics, where the thermal expansion of a solid may be thought of as the outcome of the Rayleigh radiation pressure.

IV. CONCLUSION

In this article, we have promoted the idea that introducing the Lagrange picture of fluid dynamics could be useful in the teaching of acoustics at the undergraduate level. On the one hand, the Lagrange picture can complement the Euler picture, as our alternative derivation of the reflection/transmission coefficients in appendix A shows. On the other hand, the Lagrange picture can be superior to the Euler picture, as our treatment of the Rayleigh radiation pressure has shown. The physical meaning of that pressure if much clearer in the Lagrange picture.

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Appendix A: Reflection/transmission coefficients in the Lagrange picture

We go back to the question raised in the introduction and show that the Lagrange picture provides an exact calculation of the reflection/transmission coefficients. Let us consider fig. 1 again. At rest, medium 1 and medium 2 are separated by the plane $x = 0$. Choosing the rest state of the system to implement the Lagrange labelling of the fluid elements, $x_0 = 0$ means both the right-hand face of the last slice of medium 1, and the left-hand face of the first slice of medium 2. This labelling will "follow" the motion of the system and (provided of course that no mixing occurs between the two fluids) the Lagrange labelling of the interface will remain $x_0 = 0$ throughout the propagation of the acoustic wave, whatever the amplitude of the latter and without any approximation. Let us consider an acoustic wave coming from $x = -\infty$. This incident wave is described by the displacement field $u_{\mathrm{i}}(x_0, t) = f_{\mathrm{i}}\big(t - \frac{x_0}{c_1}\big)$, where $c_1 = \sqrt{\kappa_{11}/\rho_{01}}$ is the speed of sound in medium 1 and f_{i} is any (regular) function. When the wave reaches the interface (labelled $x_0 = 0$, whatever its motion), it splits into a reflected wave $u_{\mathrm{r}}(x_0, t) = f_{\mathrm{r}}\big(t + \frac{x_0}{c_1}\big)$ and a transmitted wave $u_{\mathrm{t}}(x_0, t) = f_{\mathrm{t}}\big(t - \frac{x_0}{c_2}\big)$, where $c_2 = \sqrt{\kappa_{12}/\rho_{02}}$ is the speed of sound in medium 2. In summary, the displacement field reads

$$\begin{aligned} &\text{at } x_0 < 0: \; u(x_0,t) = f_{\mathrm{i}}\left(t - \frac{x_0}{c_1}\right) + f_{\mathrm{r}}\left(t + \frac{x_0}{c_1}\right); \\ &\text{at } x_0 > 0: \; u(x_0,t) = f_{\mathrm{t}}\left(t - \frac{x_0}{c_2}\right), \end{aligned} \tag{A1a}$$

while the (extra) pressure field reads, owing to (7):

$$\begin{aligned} &\text{at } x_0 < 0: \; p(x_0,t) = Z_1\left(f_{\mathrm{i}}'\left(t - \frac{x_0}{c_1}\right) - f_{\mathrm{r}}'\left(t + \frac{x_0}{c_1}\right)\right); \\ &\text{at } x_0 > 0: \; p(x_0,t) = Z_2 f_{\mathrm{t}}'\left(t - \frac{x_0}{c_2}\right), \end{aligned} \tag{A1b}$$

where $Z_1 = \kappa_{11}/c_1 = \sqrt{\kappa_{11}\rho_{01}}$ and $Z_2 = \kappa_{12}/c_2 = \sqrt{\kappa_{12}\rho_{02}}$ stand for the acoustic impedances of media 1 and 2. Writing the continuities of u and p at the interface $x_0 = 0$ at any time t, we obtain the well-known result

$$f_{\mathrm{r}} = \frac{Z_1 - Z_2}{Z_1 + Z_2} f_{\mathrm{i}}, \qquad f_{\mathrm{t}} = \frac{2Z_1}{Z_1 + Z_2} f_{\mathrm{i}}. \tag{A1c}$$

In the framework of the linear thermodynamic response, the Lagrange picture provides thus the simplest quantitative description of the reflection/transmission phenomenon of an acoustic wave at an interface.

Appendix B: Connection with the Grüneisen approach in solid-state physics

In section III, we have considered rigid boundary conditions, namely both pistons were fixed, or moved at a velocity imposed by the operator as regards piston L_0. One may wonder how our results would be modified if, say, piston L_0 were not fixed, but subject to the external pressure P_0. Then, the boundary condition on the fluid slice labelled L_0 would no longer be $u(x_0 = L_0) = 0$ at any time, but instead $P(x_0 = L_0) = P_0$. Of course, equations (5a) through

(10a) would hold unchanged, whereas the wavevector quantification relation (10b) would become $k_n = \left(n + \frac{1}{2}\right)\frac{\pi}{L_0}$. As regards the energy balance, (11a) and (11b) would be unchanged while energy E should be replaced by the (conserved) quantity $E + P_0 S u(L_0, t)$ in (11c) and (11d). Due to the new boundary condition, piston L_0 would move, and its position become $L_0 + \delta L(t)$. The static part $\delta L_\mathrm{s} = \langle \delta L(t) \rangle$ can be obtained by setting $P_0 = P_\mathrm{eq}(\delta L_\mathrm{s}) + \frac{1}{8}\kappa_2 A^2 k^2$, as suggested by (21), with $P_\mathrm{eq}(\delta L_\mathrm{s})$ given by (16c). Hence an acoustically induced static expansion $\frac{\delta L_\mathrm{s}}{L_0}$ of the medium ensues, given in our second-order approximation by

$$\frac{\delta L_\mathrm{s}}{L_0} = \frac{1}{8}\frac{\kappa_2}{\kappa_1} A^2 k^2 = \frac{1}{\kappa_1}(P_\mathrm{s}^{[2]} - P_0). \tag{B1}$$

Now, let us suppose that not only *one*, but *all modes*, are excited simultaneously. Averaging out all interference terms, the static relation (22b) becomes

$$P_\mathrm{s}^{[2]} - P_0 = \frac{1}{2}\frac{\kappa_2}{\kappa_1} \sum_n \mathcal{E}_n, \tag{B2a}$$

where $\mathcal{E}_n$ stands for the overall acoustic energy density in mode n (see (11d) for instance). Relation (B1) can be generalized in the same way, yielding the static expansion caused by the acoustic radiation pressure, namely

$$\frac{\delta L_\mathrm{s}}{L_0} = \frac{1}{2}\frac{\kappa_2}{\kappa_1^2} \sum_n \mathcal{E}_n. \tag{B2b}$$

We now come to the Grüneisen approach of the thermal expansion of matter in solid-state physics[18]. This approach consists in regarding solids – which are *de facto* nonlinear compounds – as linear media with volume-dependent vibrational mode frequencies. That is to say, the angular frequency of mode n is $\omega_n(L)$ and the so-called Grüneisen parameter g_n of a one-dimensional solid is defined as

$$g_n = -\frac{\mathrm{d}\ln\omega_n}{\mathrm{d}\ln L}. \tag{B3}$$

Thermal expansion originates precisely in g_n being nonzero. Now, in the calculation developed in the present paper, we find (up to the second order, see (16d) or (26a))

$$\omega_0^2 = \omega_{n0}^2\left(1 - \frac{\kappa_2}{\kappa_1}\frac{\delta L}{L_0}\right), \tag{B4a}$$

whence

$$g_n = -\frac{\delta\ln\omega_n}{\delta\ln L} = \frac{1}{2}\frac{\kappa_2}{\kappa_1}. \tag{B4b}$$

In other words, the existence of a Rayleigh acoustic radiation pressure, on the one hand, and the thermal expansion under constant P_0, on the other hand, originate in a non-vanishing Grüneisen parameter $g = \frac{1}{2}\frac{\kappa_2}{\kappa_1}$, regardless of n. In a pictorial parlance, one may say that the thermal expansion of a solid is just the outcome of the Rayleigh radiation pressure associated with the thermally excited vibrational modes.

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[19] We shall neglect viscosity throughout the present paper for the sake of simplicity.

[20] The terms "Euler picture" and "Lagrange picture" are the exact transposition to fluid mechanics of the so-called "Schrödinger picture" and "Heisenberg picture" of quantum mechanics.

[21] Let us recall that, in the Lagrange picture, the slice $[x_0, x_0 + \mathrm{d}x_0]$ does constitute a thermodynamically speaking *closed* system: no particle exchange with the outside occurs, except for a possible matter diffusion, neglected in this paper.

[22] This is justified as soon as the fluid particle velocity is small compared to the wave phase velocity. This assumption is not needed in the Lagrange picture.

[23] The reduced Planck constant $\hbar$ is introduced here artificially, in order to reckon vibrational quanta $\hbar\omega_n$ in energy units, with N_n a dimensionless parameter. Of course, since our problem has nothing to do with quantum mechanics, the numerical value of $\hbar$ used in the definition of N_n has no relevance in our reasoning.

[24] We deliberately choose this term instead of "Doppler shift" which, in the framework of the present thought experiment, is inappropriate.

[25] It is unchanged because our calculation is performed in the Lagrange picture, of course.